

**SPECTRUM OF WEIGHTED COMPOSITION
OPERATORS
PART II
WEIGHTED COMPOSITION OPERATORS ON
SUBSPACES OF BANACH LATTICES**

A. K. KITOVER

ABSTRACT. We describe the spectrum of weighted d -isomorphisms of Banach lattices restricted on closed subspaces that are “rich” enough to preserve some “memory” of the order structure of the original lattice. The examples include (but are not limited to) weighted isometries of Hardy spaces on the polydisk and unit ball in \mathbb{C}^n .

This paper is a continuation of the study of spectrum of weighted composition operators attempted in [2, Part 3] and [9]. The results of the current paper heavily depend on those in [2] and [9], and the reader is referred to [9] for the notations not explained here. The main goal is to establish a connection between the spectrum of a weighted d -isomorphism T on a Banach lattice X and the spectrum of its restriction on a closed T -invariant subspace Y of X . Surely, for such a connection to exist and be meaningful we must assume some “richness” of the subspace Y . The reader is referred to Definitions 0.2 and 0.18 for details but the Hardy spaces H^p considered as subspaces of the corresponding L^p spaces on the unit circle, or a unital uniform algebra as a subspace of the space of all continuous functions on its Shilov boundary represent typical examples of “rich” subspaces.

The following notations will be used throughout the paper.

X - a Banach lattice over the field of complex numbers \mathbb{C} .

$Z(X)$ - the center of the Banach lattice X .

$\sigma(T, Y)$ - the spectrum of a bounded linear operator T on a Banach space Y .

$\sigma_r(T, Y) = \{\lambda \in \sigma(T, Y) : \exists c > 0 \|\lambda x - Tx\| \geq c\|x\|, x \in X\}$.

$\sigma_{ap}(T, Y) = \sigma(T, Y) \setminus \sigma_r(T, Y)$.

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$\rho(T, Y)$ - spectral radius of T on Y .

Disjointness preserving operator - a linear operator T on a Banach lattice X such that $|x_1| \wedge |x_2| = 0 \Rightarrow |Tx_1| \wedge |Tx_2| = 0$, $x_1, x_2 \in X$.

d-isomorphism - a linear, bounded, invertible, and disjointness preserving operator on X . It is well known (see e.g [3]) that the inverse of such an operator also preserves disjointness.¹

The following definition was introduced by Glicksberg in [7, Definition 4.8.1]

Definition 0.1. Let A be a closed subspace of $C(K)$. We say that A is *approximating in modulus* if for any nonnegative $f \in C(K)$ and for any positive ε there is a $g \in A$ such that $\|f - |g|\|_{C(K)} \leq \varepsilon$.

For our purposes it will be useful to extend the previous definition on the class of Banach lattices.

Definition 0.2. Let X be a Banach lattice and Y be a closed subspace of X . We will say that Y is an *approximating in modulus*, or *AIM*-subspace of X if for any $x \in X$ and any positive ε there is a $y \in Y$ such that $\||x| - |y|\| \leq \varepsilon$.

Theorem 0.3. Let X be a Dedekind complete Banach lattice, S be a *d-isomorphism* of X , and $A \in Z(X)$. Let $T = AS$. Let Y be an *AIM*-subspace of X such that $TY \subseteq Y$. Then $\sigma_{ap}(T, X) = \sigma_{ap}(T, Y)$.

Proof. We start with recalling a few pretty obvious things. First notice that the *d-isomorphism* S generates an isomorphism of the center $Z(X)$ of X according to the formula $f \rightarrow Sfs^{-1}$ (see e.g [3, Proposition 8.3], [1], and [4]). To this isomorphism corresponds a homeomorphism φ of the Stonean compact K of X (we identify $Z(X)$ and $C(K)$) and $\text{supp } Sx = \varphi^{-1}(\text{supp } x)$, $x \in X$ whence

$$\text{supp } (Tx) \subseteq \varphi^{-1}(\text{supp } x) \quad (1)$$

For $f \in Z(X)$ we will identify the central operator SfS^{-1} and the function $f \circ \varphi$ in $C(K)$. We will need the formula

$$T^k = A_k S^k, k \in \mathbb{N}$$

where $A_k = A(A \circ \varphi) \dots (A \circ \varphi^{k-1})$. Recall also that the operator T is regular and that $\|Tx\| = \||T||x|\|$, $x \in X$. Let us proceed with the proof of the theorem.

The inclusion $\sigma_{ap}(T, Y) \subseteq \sigma_{ap}(T, X)$ is obvious. Let $\lambda \in \sigma_{ap}(T, X)$. Let us first consider the case when $\lambda = 0$. Let $x_n \in X$, $\|x_n\| = 1$, $Tx_n \rightarrow$

¹Often another definition of a *d-isomorphism* is used when instead of invertibility only the equivalence $|x_1| \wedge |x_2| = 0 \Leftrightarrow |Tx_1| \wedge |Tx_2| = 0$ is required.

0. Then $\|T\|x_n\| = \|Tx_n\| \rightarrow 0$. Let $y_n \in Y$ be such that $|y_n| - |x_n| \rightarrow 0$. Then $\|Ty_n\| = \|T\|y_n\| \rightarrow 0$ whence $0 \in \sigma_{ap}(T, Y)$. Assume now that $\lambda \neq 0$. Without loss of generality we can assume that $\lambda = 1$. Let $x_n \in X$, $\|x_n\| = 1$, and $Tx_n - x_n \rightarrow 0$. For any natural m let G_m be the subset of K defined as

$$G_m = \{k \in K : \varphi^m(k) = k, \varphi^j(k) \neq k, j \in \mathbb{N}, j < m.\}$$

By Frolik's theorem [17, Theorem 6.25, p.150] G_m is a clopen (maybe empty) subset of K . We will denote the corresponding band projection in X by P_m . Let also $F_m = \bigcup_{j=1}^m G_m$. It follows from the definition of

G_m and from (1) that $P_m T = T P_m$ and that the operator $T^m P_m = P_m T^m P_m$ is a band preserving and therefore a central operator on the band $P_m X$. We have to consider two possibilities.

(1) There is an $m \in \mathbb{N}$ such that $\|P_m x_n\| \xrightarrow{n \rightarrow \infty} 0$. Then $1 \in \sigma_{ap}(T^m P_m)$. Indeed, $T^m P_m x_n - P_m x_n = P_m T^m x_n - P_m x_n \xrightarrow{n \rightarrow \infty} 0$. The operator $T^m P_m$ on $P_m X$ can be represented as an operator of multiplication on $M \in C(G_m)$ and therefore there is a point $g \in G_m$ such that $M(g) = 1$.

Let us fix a positive ε and let $\delta = \left((2 + \|T\|) \left(1 + \sum_{i=1}^{m-1} \|T^i\| \right) \right)^{-1}$. Let V be a clopen neighborhood of g in G_m such that $|M - 1| < \delta$ on V and the sets $\varphi^{-j}(V)$, $j = 0, 1, \dots, m-1$ are pairwise disjoint. Because Y is an AIM-subspace of X we can find a $y \in Y$ such that $\|y\| = 1$ and $\|y - P_V y\| < \delta$ where P_V is the band projection corresponding to the clopen set V . Indeed, let x be a positive element from $P_V X$ such that $\|x\| = 1$. Let γ be such that $\frac{2\gamma}{1-\gamma} = \delta$. Let $z \in Y$ and $\|z - x\| < \gamma$. Then $\|P_V z - x\| = \|P_V z - P_V x\| < \gamma$ whence $\|z - P_V z\| < 2\gamma$. But P_V is a band projection and therefore $|z - P_V z| = |z| - |P_V z|$ whence $\|z - P_V z\| < 2\gamma$. Take $y = \frac{z}{\|z\|}$. Then $\|y - P_V y\| < \frac{2\gamma}{\|z\|} \leq \frac{2\gamma}{1-\gamma} = \delta$.

Let $v = \sum_{j=0}^{m-1} T^j y$ and $w = P_V \sum_{j=0}^{m-1} T^j y = \sum_{j=0}^{m-1} T^j P_V y$. Then it is easy to see that

- (1) $\|v - w\| < \delta \left(1 + \sum_{i=1}^{m-1} \|T^i\| \right)$.
- (2) $\|w\| \geq \|P_V y\| \geq 1 - \delta$.
- (3) $\|Tw - w\| = \|T^m P_V y - y\| = \|(M - 1)P_V y\| < \delta$.

It follows immediately from inequalities (1) – (3) that

$$\|v\| > 1 - \delta \left(2 + \sum_{i=1}^{m-1} \|T^i\| \right) > 1 - \varepsilon$$

and that

$$\|Tv-v\| \leq \|v-w\| + \|Tv-Tw\| + \|Tw-w\| < \delta(2+\|T\|)\left(1+\sum_{i=1}^{m-1}\|T^i\|\right) = \varepsilon.$$

Because ε can be made arbitrary small we see that $1 \in \sigma_{ap}(T, Y)$.

(2) For any natural m we have $\|P_m x_n\| \xrightarrow{n \rightarrow \infty} 0$. Let us fix a large natural N and let $K_N = K \setminus F_{2N+1}$. Let Q be the band projection in X corresponding to the clopen set K_N and let $z_n = Qx_n$. Then $\|z_n\| \xrightarrow{n \rightarrow \infty} 1$ and $Tz_n - z_n \xrightarrow{n \rightarrow \infty} 0$. It follows from the fact that K is extremally disconnected, from the definition of the set K_N , and from the Zorn's lemma that there is a maximal by inclusion clopen subset E of K_N such that

$$\varphi^{-i}(E) \cap \varphi^{-j}(E) = \emptyset, \quad 0 \leq i < j \leq 2N. \quad (2)$$

Let P_E be the band projection corresponding to the clopen set E . We claim that

$$\liminf_{n \rightarrow \infty} \|P_E z_n\| > 0. \quad (3)$$

Indeed, assume to the contrary that $\liminf_{n \rightarrow \infty} \|P_E z_n\| = 0$. By choosing if necessary a subsequence we can assume that $\lim_{n \rightarrow \infty} \|P_E z_n\| = 0$. Let $E_i = E \cap \varphi^i(E)$, $i \in [2N+2 : 4N+1]$. It follows from the fact that E is a maximal by inclusion clopen subset of K_N satisfying (2) that the sets $\varphi^{-j}(E_i)$, $2N+2 \leq i \leq 4N+1$, $0 \leq j \leq i-1$ make a partition of K_N (the proof is basically the same as the one of the Halmos - Rokhlin lemma). Let P_{ij} be the band projection corresponding to the set $\varphi^{-j}(E_i)$. Then $P_{ij}T^i z_n = (P_{i0} \circ \varphi^i)T^i z_n = T^i P_{i0} z_n$. But P_{i0} is the band projection corresponding to the set E_i whence $\|P_{i0} z_n\| \leq \|P_E z_n\| \xrightarrow{n \rightarrow \infty} 0$ and therefore $P_{ij}T^i z_n \xrightarrow{n \rightarrow \infty} 0$. But $T^i z_n - z_n \xrightarrow{n \rightarrow \infty} 0$ whence $P_{ij}z_n \xrightarrow{n \rightarrow \infty} 0$. Recalling that $\sum_{i,j} P_{ij} = Q$ and that $Qz_n = z_n$ we see that $z_n \xrightarrow{n \rightarrow \infty} 0$, a contradiction.

Let $\mathcal{E} = \{E : E \text{ is maximal by inclusion closed subset of } K_N \text{ satisfying (2)}\}$. Notice that because φ is a homeomorphism we have $E \in \mathcal{E} \Leftrightarrow \varphi(E) \in \mathcal{E}$. Let $s_n = \sup_{E \in \mathcal{E}} \|P_E z_n\|$. It follows from (3) that $\liminf_{n \rightarrow \infty} s_n = s$ where $0 < s \leq 1$. Let $E_n \in \mathcal{E}$ be such that $\|(P_n \circ \varphi^N)z_n\| \geq s_n - 1/n$ where $P_n = P_{E_n}$. Then $\liminf_{n \rightarrow \infty} \|(P_n \circ \varphi^N)z_n\| = s$ and we can assume without loss of generality that $\lim_{n \rightarrow \infty} \|(P_n \circ \varphi^N)z_n\| = s$.

For any $n \in \mathbb{N}$ choose $y_n \in Y$ such that $\||y_n| - |P_n z_n|\| < 1/n$. Notice that then

$$\||P_n y_n| - |P_n z_n|\| = \|P_n(|y_n| - |P_n z_n|)\| \leq 1/n. \quad (3')$$

Let

$$g_n = \sum_{j=0}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} T^j y_n.$$

We claim that

$$\liminf_{n \rightarrow \infty} \|g_n\| \geq s \quad (4)$$

. Indeed

$$\|g_n\| \geq \|(P_n \circ \varphi^N)g_n\| = \left\| \sum_{j=0}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} (P_n \circ \varphi^N) T^j y_n \right\|.$$

Consider $i = N$ and the corresponding term in the sum above which is $(P_n \circ \varphi^N) T^N y_n$. Notice that

$$\begin{aligned} (P_n \circ \varphi^N) T^N y_n &= (P_n \circ \varphi^N) A_N S^N y_n = A_N (P_n \circ \varphi^N) S^N y_n = \\ &= A_N S^N S^{-N} (P_n \circ \varphi^N) S^N y_n = T^N P_n y_n. \end{aligned}$$

Next notice that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|T^N P_n y_n\| &= \lim_{n \rightarrow \infty} \|T^N P_n z_n\| = \\ &= \lim_{n \rightarrow \infty} \|(P_n \circ \varphi^N) T^N z_n\| = \lim_{n \rightarrow \infty} \|(P_n \circ \varphi^N) z_n\| = s. \end{aligned}$$

Now consider $j \in [0 : 2N], j \neq N$. It follows from (2) and (3') that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|(P_n \circ \varphi^N) T^j y_n\| &= \lim_{n \rightarrow \infty} \|(P_n \circ \varphi^N) T^j P_n y_n\| = \\ &= \lim_{n \rightarrow \infty} \|(P_n \circ \varphi^N)(P_n \circ \varphi^j) T^j y_n\| = 0, \end{aligned}$$

and (4) is proved.

On the other hand

$$\begin{aligned} Tg_n - g_n &= -\left(1 - \frac{1}{\sqrt{N}}\right)^N y_n - \frac{1}{\sqrt{N}} \sum_{j=1}^N \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} T^j y_n \\ &\quad + \frac{1}{\sqrt{N}} \sum_{j=N+1}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{j-N} T^j y_n + \left(1 - \frac{1}{\sqrt{N}}\right)^N T^{2N+1} y_n \quad (5) \end{aligned}$$

Let $p_n = \sum_{j=1}^N \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} T^j y_n$ and $q_n = \sum_{j=N+1}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{N-j} T^j y_n$. The sequences p_n and q_n are norm bounded and we claim that

$$\|p_n + q_n\| - \|p_n - q_n\| \xrightarrow{n \rightarrow \infty} 0 \quad (6)$$

. To prove (6) it is enough to prove that $|p_n| \wedge |q_n| \xrightarrow{n \rightarrow \infty} 0$. But

$$|p_n| \leq \sum_{j=1}^N \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |y_n|, \quad |q_n| \leq \sum_{j=N+1}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |y_n|,$$

$$\sum_{j=1}^N \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |y_n| - \sum_{j=1}^N \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |P_n z_n| \xrightarrow{n \rightarrow \infty} 0, \quad \sum_{j=N+1}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |y_n| - \sum_{j=N+1}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |P_n z_n| \xrightarrow{n \rightarrow \infty} 0, \text{ and in view of}$$

$$(2)$$

$$\sum_{j=1}^N \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |P_n z_n| \wedge \sum_{j=N+1}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} |T|^j |P_n z_n| = 0.$$

and (6) is proved.

$$\text{Let } h_n = -\left(1 - \frac{1}{\sqrt{N}}\right)^N y_n + \frac{1}{\sqrt{N}} \sum_{j=1}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j|} T^j y_n + \left(1 - \frac{1}{\sqrt{N}}\right)^N T^{2N+1} y_n.$$

It follows from (6) that $\|Tg_n - g_n\| - \|h_n\| \rightarrow 0$. Notice that $h_n = \frac{1}{\sqrt{N}}g_n - \left(1 - \frac{1}{\sqrt{N}}\right)^{N+1}y_n + \left(1 - \frac{1}{\sqrt{N}}\right)^N T^{2N+1}y_n$, that $\limsup \|y_n\| = \limsup \|P_n z_n\| = s$, and that $\limsup \|T^{2N+1}y_n\| = \limsup \|T^{2N+1}P_n z_n\| = \limsup \|(P_n \circ \varphi^{2N+1})T^{2N+1}z_n\| = \limsup \|(P_n \circ \varphi^{2N+1})z_n\| \leq s$ (we used here that $T^{2N+1}z_n - z_n \rightarrow 0$ and that $\varphi^{-(2N+1)}(E) \in \mathcal{E}$). Finally we get that

$$\limsup \|Tg_n - g_n\| \leq \left[\frac{1}{\sqrt{N}} + 2\left(1 - \frac{1}{\sqrt{N}}\right)^N \right] \liminf \|g_n\|.$$

Because N can be chosen arbitrary large we see that $1 \in \sigma_{ap}(T, Y)$. \square

Corollary 0.4. *Assume conditions of Theorem 0.3 and assume additionally that the powers $T^n, n = 0, 1, \dots$ are pairwise disjoint in the lattice $L_r(X)$ of all regular operators on X . Then $\sigma(T, Y)$ is rotation invariant.*

Proof. It follows from the conditions of the corollary and Theorem 3.14 in [2] that the set $\sigma_{ap}(T, X)$ is rotation invariant whence by Theorem 0.3 $\sigma_{ap}(T, Y)$ is rotation invariant. Because the set $\sigma(T, Y) \setminus \sigma_{ap}(T, Y)$ is open in \mathbb{C} we see that $\sigma(T, Y)$ is rotation invariant. \square

The next lemma will be used in the sequel but it also might be of independent interest because it provides some information about the residual spectrum of weighted d -isomorphisms.

Lemma 0.5. *Let X be a Dedekind complete Banach lattice, S be a d -isomorphism of X , $A \in Z(X)$, $\text{supp } A = K$, and $T = AS$. Assume that $\lambda \in \sigma(T) \setminus \{0\}$ and $(\lambda I - T)X = X$ (i.e. $\lambda \in \sigma_r(T^*, X^*)$). Then there is a sequence $\{u_i\}_{i \in \mathbb{Z}}$ of nonzero elements in X such that*

- The elements u_i , $i \in \mathbb{Z}$ are pairwise disjoint in X .
- $\sum_{i=-\infty}^{\infty} \|u_i\| < \infty$.
- $Tu_i = \lambda u_{i+1}$, $i \in \mathbb{Z}$.

Proof. We can assume without loss of generality that $\lambda = 1$. Let $x \in X \setminus \{0\}$ be such that $Tx = x$. Let $E = \text{supp } x$. The condition $\text{supp } A = K$ guarantees that $\varphi(E) = E$. Let X_E be the band in X corresponding to E . Then the band projection P_E commutes with T and therefore $(I - T)X_E = X_E$ whence $1 \in \sigma_r(T^*, X_E^*)$. Because the set $\sigma_r(T^*, X_E^*)$ is open in \mathbb{C} we can find $\varepsilon > 0$ and $y \in X_E \setminus \{0\}$ such that $Ty = (1 + \varepsilon)y$. Let $F = \text{supp } y$ then $F \subseteq E$ and $\varphi(F) = F$. Applying this procedure once again (and decreasing ε , if necessary) we obtain a nonempty clopen subset H of K and elements $u, v, w \in X_H$ such that

- $\varphi(H) = H$.
- $\text{supp } u = \text{supp } v = \text{supp } w = H$.
- $Tu = u$, $Tv = (1 + \varepsilon)v$, and $Tw = (1 - \varepsilon)w$.

Notice that the set H cannot contain φ -periodic points. Indeed, otherwise using the Frolik's theorem and the fact that $Tu = u$ we would be able to find a nontrivial band B in X and $m \in \mathbb{N}$ such that $T^m|B = I|B$ whence $(T^*)^m|B^* = I|B^*$ in contradiction with our assumption that $1 \in \sigma_r(T^*)$.

Multiplying, if necessary, the elements u , v , and w by appropriate positive constants we can find a clopen subset L of H such that $P_L|u| \leq P_L|v| \leq 2P_L|u|$ and $P_L|u| \leq P_L|w|$. We claim that there is an $N \in \mathbb{N}$ such that $\forall n \geq N$ $L \cap \varphi^{-n}(L) = \emptyset$. To prove it notice that $|T^n P_L u| = |A_n S^n P_L u| = |A_n P_L S^n P_L S^{-n} S^n u| = |A_n (P_L \circ \varphi^n) S^n u| = |P_{\varphi^{-n}(L)} T^n u| = |P_{\varphi^{-n}(L)} u|$. Similarly, $|T^n P_L v| = (1 + \varepsilon)^n |P_{\varphi^{-n}(L)} v|$. Let $N \in \mathbb{N}$ be such that $(1 + \varepsilon)^N \geq 3$. Assume contrary to our claim that for some $n \geq N$ the intersection $M = L \cap \varphi^{-n}(L) \neq \emptyset$. Recalling that $|T^n x| = |T||x|$, $x \in X$, $n \in \mathbb{N}$ we see that

$$\begin{aligned} |P_M 2u| &= |P_M T^n P_L 2u| \geq |P_M T^n P_L v| = \\ &= (1 + \varepsilon)^n |P_M v| \geq 3|P_M v| \geq |P_M 3u| \end{aligned}$$

whence $P_M u = 0$ in contradiction with $M \subset \text{supp } u$. Next, because L does not contain φ periodic points we can find a clopen nonempty subset R of L such that the sets $\varphi^{-n}(R)$, $n = 0, 1, \dots, N$ are pairwise disjoint. Therefore the sets $\varphi^{-n}(R)$, $n \in \{0\} \cup \mathbb{N}$ are pairwise disjoint, and because φ is a homeomorphism the same is true for the sets $\varphi^n(R)$, $n \in \mathbb{Z}$. Let $u_i = P_{\varphi^{-i}(R)} u$, $i \in \mathbb{Z}$. Similarly we define the elements v_i and w_i . Then $Tu_i = u_{i+1}$, $Tv_i = (1 + \varepsilon)v_{i+1}$, and $Tw_i = (1 - \varepsilon)w_{i+1}$. Therefore $|u_n| = |T|^n |u_0| \leq |T|^n |w_0| = (1 - \varepsilon)^n |w_n|$ whence

$\|u_n\| \leq (1 - \varepsilon)^n \|w\|, n \in \mathbb{N}$. Next for any $n \in \mathbb{N}$ we have $T^n u_{-n} = u_0$ and $T^n v_{-n} = (1 + \varepsilon)^n v_0$ whence $|T^n u_{-n}| \leq |T^n (1 + \varepsilon)^{-n} v_{-n}|$. Because $\text{supp } A = K$ the last inequality is equivalent to $|u_{-n}| \leq (1 + \varepsilon)^{-n} |v_{-n}|$ whence $\|u_{-n}\| \leq (1 + \varepsilon)^{-n} \|v\|$. \square

Remark 0.6. (1) Because $SA = S(AS)S^{-1}$ the conclusion of Lemma 0.5 is valid for operators of the form SA .

(2) The proof of Lemma 0.5 shows that the condition $\text{supp } A = K$ can be substituted by the following one. For any μ from some open neighborhood of λ and for any nonzero $x \in X$ such that $Tx = \mu x$ we have $\varphi(\text{supp } x) = \text{supp } x$.

Theorem 0.7. *Assume conditions of Theorem 0.3. Then $\sigma(T, X) \subseteq \sigma(T, Y)$.*

Proof. Assume to the contrary that there is $\lambda \in \sigma(T, X) \setminus \sigma(T, Y)$. By Theorem 0.3 $\lambda \in \sigma_r(T, X)$ and therefore we can assume without loss of generality that $\lambda = 1$. Then $\sigma_{ap}(T, X) \cap \Gamma \neq \Gamma$ and it follows from [2, Proposition 12.15] that X is the direct sum of two disjoint T -invariant bands X_1 and X_2 (band X_1 might be empty) such that some power of T restricted on X_1 is a multiplication operator and $\Gamma \subset \sigma_r(T, X_2)$. Consider the subspace Y_2 of X defined as $Y_2 = cl \{P_2 y : y \in Y\}$. Clearly $TY_2 \subseteq Y_2$. It is easy to see that Y_2 is an AIM subspace of X_2 . Indeed, let $x \in X_2$ and let $y_n \in Y$ be such that $|y_n| \rightarrow |x|$. Then $|P_2 y_n| \rightarrow |P_2 x| = |x|$. Moreover, $\Gamma \cap \sigma(T, Y_2) = \emptyset$. To see it notice that for any $\gamma \in \Gamma$ the operator $\gamma I - T$ is bounded from below on Y_2 because $\gamma \in \sigma_r(T, X_2)$. On the other hand consider $y \in Y$. Because the operator $I - T$ is invertible on Y there is $z \in Y$ such that $z - Tz = y$ whence $P_2 z - TP_2 z = P_2 y$. Thus the operator $(I - T)|_{Y_2}$ is bounded from below and its image contains a dense subspace of Y_2 whence it is invertible on Y_2 . It remains to notice that because the resolvent set and the residual spectrum of a bounded operator are open in \mathbb{C} the operator $(\gamma I - T)|_{Y_2}$ is invertible for any $\gamma \in \Gamma$.

The previous paragraph shows that without loss of generality we can assume that $\Gamma \subset \sigma_r(T, X)$ and $\Gamma \cap \sigma(T, Y) = \emptyset$. Then $\Gamma \subset \sigma_r(T^{**})$. Assume first that for any μ in some open neighborhood of 1 and for any nonzero $u \in X^*$ such that $T^*u = \mu u$ we have $\psi(\text{supp } u) = \text{supp } u$ where ψ is the homeomorphism of the Stonean space Q of X^* generated by the isomorphism $f \rightarrow (S^*)^{-1}fS^*$, $f \in Z(X^*)$. Then by Lemma 0.5 and Remark 0.6 there are pairwise disjoint elements $u_i \in X^*, i \in \mathbb{Z}$ such that $u_0 \neq 0$, $T^*u_i = u_{i+1}, i \in \mathbb{Z}$, and $\sum_{i=-\infty}^{\infty} \|u_i\| < \infty$. Let $y \in Y$, then $|y| = fy$ where $f \in Z(X)$ and $|f| \equiv 1$. Let $F = f^* \in Z(X^*)$ and let $v_i = (S^*)^i F (S^*)^{-i} u_i$. it is immediate to see that $\|v_i\| = \|u_i\|$

and that $T^*v_i = v_{i+1}, i \in \mathbb{Z}$. For any $\gamma \in \Gamma$ let $w_\gamma = \sum_{i=-\infty}^{\infty} \gamma^{-i} v_i$. Then $T^*w_\gamma = \gamma w_\gamma$ and therefore (because $\sigma(T, Y) \cap \Gamma = \emptyset$) we have

$$\forall \gamma \in \Gamma \quad \langle y, w_\gamma \rangle = \sum_{i=-\infty}^{\infty} \gamma^{-i} \langle y, v_i \rangle = 0.$$

Therefore

$$\langle |y|, u_0 \rangle = \langle y, v_0 \rangle = \int_0^{2\pi} \left(\sum_{i=-\infty}^{\infty} e^{-i\theta} \langle y, v_i \rangle \right) d\theta = 0.$$

But Y is an AIM subspace of X whence $u_0 = 0$, a contradiction.

It remains to consider the case when there are $\lambda \neq 0$ and $u \in X^* \setminus \{0\}$ such that $\lambda \notin \sigma(T, Y)$, $T^*u = \lambda u$, and $E \subsetneq \psi^{-1}(E)$ where $E = \text{supp } u$. Notice that $s^*A^*(S^*)^{-1} \equiv 0$ on $\psi^{-1}(E) \setminus E$. Without loss of generality we can assume that $\lambda = 1$. Let $F = E \setminus \psi(E)$. Notice that $\psi^i(F) \cap \psi^j(F) = \emptyset, i \neq j, i, j \in \mathbb{Z}$. Let $u_n = P_n u, n \in \mathbb{N}$, where P_n is the band projection corresponding to the set $\psi^n(F)$. Then $T^*u_n = u_{n-1}, n \in \mathbb{N}$, $u_0 \neq 0$, and $T^*u_0 = 0$. Let $y \in Y$ and introduce $v_n, n = 0, 1, \dots$ like in the previous part of the proof. For any $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$ the series $w(\alpha) = \sum_{n=0}^{\infty} \alpha^n v_n$ converges in norm and $T^*w(\alpha) = \alpha w(\alpha)$.

The function $W(\alpha) = \langle y, w(\alpha) \rangle$ is analytic in the open unit disk and identically zero on the intersection of the unit disk and the resolvent set of $T|Y$ whence it is identically zero and $\langle |y|, u_0 \rangle = \langle y, v_0 \rangle = 0$. We conclude that $u_0 = 0$, a contradiction. \square

Definition 0.8. Let X be a Banach lattice and Y be a closed subspace of X . We will say that Y is an analytic subspace of X if for any nonzero band E in X the implication holds $y \in Y, y \perp E \Rightarrow y = 0$.

Theorem 0.9. Let Y be an analytic AIM subspace of a Banach lattice X . Let S be a d -isomorphism of X , $A \in Z(X)$, $T = AS$, and $TY \subseteq Y$. Then the set $|\sigma(T, Y)| = \{|\lambda| : \lambda \in \sigma(T, Y)\}$ is a connected subset of \mathbb{R} .

Proof. Assume to the contrary that there is a positive r such that $\sigma(T, Y) = \sigma_1 \cup \sigma_2$ where σ_1 and σ_2 are nonempty subsets of \mathbb{C} , $\sigma_1 \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and $\sigma_2 \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$. Let Y_1 and Y_2 be the corresponding spectral subspaces of the operator $T|Y$. By Theorem 0.7 $\sigma(T, X) \cap r\Gamma = \emptyset$. Let X_1 and X_2 be the corresponding spectral subspaces. Then clearly $Y_i \subseteq X_i, i = 1, 2$. It follows from [2,

Theorem 13.1] that ² $X_1 \perp X_2$ whence $Y_1 \perp Y_2$ in contradiction with our assumption that Y is an analytic subspace of X . \square

Corollary 0.10. *Assume conditions of Corollary 0.4 and assume additionally that Y is an analytic subspace of X . Then $\sigma(T, Y)$ is either a circle or an annulus centered at 0.*

Now we will consider some examples.

Example 0.11. Let (Γ, m) be the unit circle with the normalized Lebesgue measure m . Let X be a Dedekind complete Banach lattice such that $L^\infty(m) \subseteq X \subseteq L^1(m)$. Let $Y = \{x \in X : \int_0^{2\pi} x(e^{i\theta})e^{in\theta}dm = 0, n \in \mathbb{N}\}$. Let T be a d -isomorphism of X such that $TY \subseteq Y$ and the powers T^n , $n = 0, 1, \dots$ are pairwise disjoint. Then $\sigma(T, Y)$ is either a circle or an annulus centered at 0.

Proof. First notice that Y is a closed subspace of X because $X \subseteq L^1(m)$. To prove that Y is an AIM of X let us consider $x \in X$ and the sequence $z_n = |x| + \frac{1}{n}\mathbf{1}$, $n \in \mathbb{N}$ where $\mathbf{1}$ is the constant function identically equal to 1. Then, because $L^\infty(m) \subseteq X$ we have $z_n \in X$ and $z_n \xrightarrow{X} |x|$. Notice now that $\ln z_n \in L^1(m)$ and therefore (see e.g. [8, page 53]) there are $h_n \in H^1$ such that $|h_n| = z_n$. Then $h_n \in Y$ and therefore Y is an AIM subspace of X . It remains to notice that Y is an analytic subspace of X , because $Y \subseteq H^1$, and apply Corollary 0.10. \square

Example 0.12. Let $p \in \mathbb{N}$. Let Ω be either (Γ^p, m) - the p -dimensional unit torus with the Haar measure m or (B_p, m) the unit ball in \mathbb{C}^p with the normalized Lebesgue measure m (see [14, 1.4.1]). Let X be a Banach lattice with order continuous norm such that $L^\infty(\Omega) \subseteq X \subseteq L^1(\Omega)$. Let $Y = X \cap H^1(\Omega)$. Let T be a d -isomorphism of X such that $TY \subseteq Y$ and the powers T^n , $n = 0, 1, \dots$ are pairwise disjoint. Then $\sigma(T, Y)$ is either a circle or an annulus centered at 0.

Proof. Let $x \in X$. Because the norm in X is order continuous $|x|$ can be approximated by norm in X by strictly positive functions z_n from $C(\Omega)$. There are $h_n \in H^\infty(\Omega)$ such that $|h_n| = z_n$. In the case of polydisc it follows from Theorem 3.5.3 in [13] and in the case of the unit ball in \mathbb{C}^p from the deep results of Alexandrov [5] and Løw [10] (see also [15]). The rest of the proof goes like in Example 0.11. \square

²Strictly speaking we should apply Theorem 13.1 from [2] to the operator T^{**} but it provides the desired result.

The condition that a subspace of a Banach lattice is *AIM* is quite restrictive and in the following part of the paper we will weaken it but at the price of imposing additional conditions either on the space X or on the operator T .

Recall that if X is an arbitrary Banach lattice and T is an order continuous operator on X then T can be extended in the unique way on the Dedekind completion \hat{X} of X (see [16]).

Theorem 0.13. *Let X be a Banach lattice, S be a d -isomorphism of X , $A \in Z(X)$, and $T = AS$. Let \hat{X} be the Dedekind completion of X and $\hat{T} = \hat{A}\hat{S}$ be (the unique) extension of T on \hat{X} . Assume also that \hat{X} has the weak sequential Fatou property. Then $\sigma_{ap}(T, X) = \sigma_{ap}(\hat{T}, \hat{X})$.*

Proof. Assume first that \hat{X} has the sequential Fatou property. The inclusion $\sigma_{ap}(T, X) \subseteq \sigma_{ap}(\hat{T}, \hat{X})$ is trivial. If $0 \in \sigma_{ap}(\hat{T}, \hat{X})$ then \hat{A} is not invertible in $Z(\hat{X})$ and it is routine to see that $0 \in \sigma_{ap}(T, X)$. Thus it is enough to prove that if $1 \in \sigma_{ap}(\hat{T}, \hat{X})$ then $1 \in \sigma_{ap}(T, X)$. We will use the notations from the proof of Theorem 0.3. Consider $x_n \in \hat{X}$ such that $\|x_n\| = 1$ and $Tx_n - x_n \rightarrow 0$. Assume that there is a natural m such that $\|P_m x_n\| \not\rightarrow_{n \rightarrow \infty} 0$. Then there is a point $t \in G_m$ such that $\hat{A}_m(t) = 1$. For every $k \in \mathbb{N}$ we can find a clopen subset V_k of G_m such that the sets $V_k, \varphi^{-1}(V_k), \dots, \varphi^{-(m-1)}(V_k)$ are pairwise disjoint and $|\hat{A}_m(k) - 1| < 1/k, k \in V_k$. Let $x_k \in X$ be such that $\|x_k\| = 1$ and x_k belongs to the band in X corresponding to the set V_k . Let $y_k = \sum_{i=0}^{m-1} T^i x_k$. Then it is immediate to see that $\|Ty_k - y_k\| = o(\|y_k\|), k \rightarrow \infty$.

Now assume that $P_m x_n \rightarrow_{n \rightarrow \infty} 0$ for any $m \in \mathbb{N}$. Let N, z_n, E_n , and P_n be as in the proof of Theorem 0.3. Let $u_n = P_n z_n$ and

$$g_n = \sum_{j=0}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} T^j u_n.$$

Because \hat{X} has the Fatou property for every $n \in \mathbb{N}$ there is $v_n \in X$ such that $|v_n| \leq |u_n|$ and $\|v_n\| > \|u_n\| - 1/n$. Let

$$h_n = \sum_{j=0}^{2N} \left(1 - \frac{1}{\sqrt{N}}\right)^{|j-N|} T^j v_n.$$

Then like in the proof of Theorem 0.3 we can see that

$$\limsup \|Th_n - h_n\| \leq \left[\frac{1}{\sqrt{N}} + 2\left(1 - \frac{1}{\sqrt{N}}\right)^N\right] \liminf \|h_n\|.$$

Assume now that \hat{X} has only the weak sequential Fatou property. Then the formula $|||x||| = \inf_{n \rightarrow \infty} \lim \|x_n\|$, where the infimum is taken over all the sequences $\{x_n\}$ of elements of \hat{X} such that $|x_n| \uparrow |x|$, defines an equivalent norm on \hat{X} with the Fatou property. Because changing a norm to an equivalent one does not change the spectrum of a bounded linear operator we are done. \square

Theorem 0.14. *Assume conditions of Theorem 0.13. Then $\sigma(\hat{T}, \hat{X}) \subseteq \sigma(T, X)$.*

Proof. Assume to the contrary that there is $\lambda \in \sigma(\hat{T}, \hat{X}) \setminus \sigma(T, X)$. Then by Theorem 0.13 $\lambda \in \sigma_r(\hat{T}, \hat{X})$. We can assume that $\lambda = 1$. Then (see [2, Chapter 13]) there are two possibilities.

(1) $\Gamma \cap \sigma(T, X) = \emptyset$. Then by Proposition 13.3 in [2] X is the direct sum of two disjoint T -invariant bands X_1 and X_2 such that $\sigma(T, X_1) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\sigma(T, X_2) \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$. Then \hat{X} is the direct sum of disjoint \hat{T} -invariant bands \hat{X}_1 and \hat{X}_2 and $\rho(\hat{T}|_{\hat{X}_1}) = \rho(T, X_1) < 1$. The restrictions $T|_{X_2}$ and $\hat{T}|_{\hat{X}_2}$ are a d -isomorphisms on X_2 and respectively \hat{X}_2 . Let $R : X_2 \rightarrow X_2$ be the inverse of $T|_{X_2}$. Then \hat{R} is the inverse of $\hat{T}|_{\hat{X}_2}$ and $\rho(R) = \rho(\hat{R}) < 1$. Therefore $\sigma(\hat{T}, \hat{X}) \cap \Gamma = \emptyset$, a contradiction.

(2) Assume now that $\sigma(\hat{T}, \hat{X}) \cap \Gamma \neq \emptyset$. Let K be the Stonean compact of \hat{X} and φ be the homeomorphism of K generated by the operator \hat{S} . For any $m \in \mathbb{N}$ let F_m be the clopen subset of K that consists of all φ -periodic points of period $\leq m$, and let \hat{B}_m be the band in \hat{X} corresponding to F_m . Let also $B_m = X \cap \hat{B}_m$. Notice that for any m the bands B_m and \hat{B}_m are invariant for T and respectively \hat{T} . Let \dot{T}_m and $\dot{\hat{T}}_m$ be the factor operators on X/B_m and \hat{X}/\hat{B}_m , respectively. Then it follows from Proposition 12.15 in [2] that there is such $m \in \mathbb{N}$ that $\sigma(\dot{T}_m) \cap \Gamma = \emptyset$ and $\Gamma \subset \sigma_r(\dot{\hat{T}}_m)$. Because $\widehat{X/B_m} = \hat{X}/\hat{B}_m$ we are now in the conditions of part (1) of the proof and come to a contradiction. \square

Remark 0.15. It is not known to the author whether the condition that \hat{X} has the weak Fatou property in Theorems 0.13 and 0.14 can be dropped.

In connection with Theorems 0.13 and 0.14 it might be interesting to obtain a criterion for a point in the spectrum of a weighted d -isomorphism $T = AS$ to belong either to $\sigma_{ap}(T)$ or to $\sigma_r(T)$. A complete answer at the present is known only under additional condition that $\sigma(S) \subseteq \Gamma$ (see Theorem 0.19 below and the results that follow it). Nevertheless, Lemma 0.5 allows us to obtain the following result.

Theorem 0.16. *Let X be a Banach lattice with order continuous norm, S be a d -isomorphism of X , $A \in Z(X)$, and $T = AS$. Let $\lambda \in \sigma_r(T)$.*

Then

(1) *There is a nonzero band E in X such that the bands $S^i E, i \in \mathbb{Z}$ are pairwise disjoint.*

(2) *X is the union of four disjoint T -invariant bands F_1, F_2, F_3 , and F_4 (any of the bands F_2, F_3 , and F_4 can be 0) such that*

$$F_1 = \left\{ \bigcup_{i=-\infty}^{\infty} S^i E \right\}^{dd}.$$

$$\sigma(T|F_2) \subset \{\alpha \in \mathbb{C} : |\alpha| < |\lambda|\}.$$

$$\sigma(T|F_3) \subset \{\alpha \in \mathbb{C} : |\alpha| > |\lambda|\}.$$

Some power of the operator $T|F_4$ is a multiplication operator and $\lambda \notin \sigma(T|F_4)$.

Proof. First notice that (see e.g. [12, Theorem 2.4.2.]) that X is Dedekind complete and its canonical image in the second dual X^{**} is an ideal in X^{**} . It follows from Lemma 0.5 and the proof of Theorem 0.7 that there is a nonzero band H in X^* such that the bands $(S^*)^i, i \in \mathbb{Z}$ are pairwise disjoint. Let H_1 be a maximal by inclusion band in X^* with the same property (it exists because X^* is Dedekind complete) let $L_1 = \left\{ \bigcup_{i=-\infty}^{\infty} (S^*)^i H_1 \right\}^{dd}$, and let L be the disjoint complement of L_1 in X^* . Then $\lambda \notin \sigma(T^*|L)$ and the set $\lambda\Gamma \cap \sigma(T^*|L)$ is at most finite. Therefore L is the union of three disjoint T^* -invariant bands L_2, L_3 , and L_4 such that $\sigma(T^*|L_2) \subset \{\alpha \in \mathbb{C} : |\alpha| < |\lambda|\}$, $\sigma(T^*|L_3) \subset \{\alpha \in \mathbb{C} : |\alpha| > |\lambda|\}$, and some power of the operator $T^*|L_4$ is a multiplication operator and $\lambda \notin \sigma(T^*|L_4)$.

To finish the proof it remains to identify X with its canonical image in X^{**} and take $F_i = (L_i)^* \cap X, i = 1, \dots, 4$. \square

Corollary 0.17. *Assume conditions of Theorem 0.16 and assume additionally that the operator T is band irreducible and that the Banach lattice X contains no atoms. Then $\sigma(T, X) = \sigma_{ap}(T, X)$.*

Similarly to Remark 0.15 it is not clear whether the condition that X has order continuous norm in Theorem 0.16 or Corollary 0.17 can be weakened.

The proof of Theorem 0.13 heavily depends on the fact that for any $x \in \hat{X}$ there is a $y \in X$ such that $|y| \leq |x|$. Of course we cannot expect such a property to be true if we consider analytic subspaces of Banach lattices; nevertheless often we can assume a close property that is described in the following definition.

Definition 0.18. Let X be a Banach lattice and Y be a closed subspace of X . We say that Y is *almost localized* in X if for any band Z in X and for any $\varepsilon > 0$ there is a $y \in Y$ such that $\|y\| = 1$ and $\|(I - P_Z)y\|_{\hat{X}} < \varepsilon$ where P_Z is the band projection on the band \hat{Z} generated by Z in the Dedekind completion \hat{X} of X .

Theorem 0.19. Let X be a Dedekind complete Banach lattice and Y be an almost localized subspace of X . Let A and U be linear bounded disjointness preserving operators on X such that

- (1) $U \geq 0$ ³ and $\sigma(U) \subseteq \Gamma$.
- (2) $A \in Z(X)$.
- (3) $AUY \subseteq Y$.

Let $T = AU$. Then $\sigma_{ap}(T, X) = \sigma_{ap}(T, Y)$.

Proof. Let K be the Stonean space of the lattice X . It is well known that the center $Z(X)$ of X can be identified with $C(K)$. By Proposition 8.4 in [3] the map $f \rightarrow UfU^{-1}$, $f \in C(K)$ is an isomorphism of algebra $C(K)$. Let φ be the homeomorphism of K corresponding to that isomorphism. We identify A with a function from $C(K)$ which will be denoted also by A and introduce an auxiliary operator S on $C(K)$ acting by the formula

$$(Sf)(k) = A(k)f(\varphi(k)).$$

We will prove that $\sigma_{ap}(S, C(K)) = \sigma_{ap}(T, X) = \sigma_{ap}(T, Y)$.

Part 1. Here we will prove that $\sigma_{ap}(S, C(K)) \subseteq \sigma_{ap}(T, Y)$.

Assume that $0 \in \sigma_{ap}(S)$. Then A is not invertible in $C(K)$. Fix a positive ε and a clopen subset E of K such that $|A| \leq \varepsilon$ on E . Let B be the band in X corresponding to E . Because Y is almost localized in X and $C = U^{-1}B$ is a band in X we can find $y \in Y$ such that $\|y\| = 1$ and $\|(I - P_C)y\| \leq \varepsilon$. Then

$$\|Ty\| \leq \|T(I - P_C)y\| + \|TP_Cy\| \leq \varepsilon\|T\| + \|TP_Cy\|.$$

and

$$\|TP_Cy\| = \|AUP_Cy\| = \|AUP_CU^{-1}Uy\| = \|AP_BUy\| \leq \varepsilon\|U\|.$$

Therefore $\|Ty\| \leq (\|T\| + \|U\|)\varepsilon$ and $0 \in \sigma_{ap}(T, Y)$.

Next assume that $\lambda \in \sigma_{ap}(S)$ and $\lambda \neq 0$. Without loss of generality we can assume that $\lambda = 1$. Notice that for $n \in \mathbb{N}$ $T^n = A_n U^n$ where

³The assumption that $U \geq 0$ does not result in loss of generality because any d -isomorphism S on a Dedekind complete Banach lattice can be represented as $M|S|$ where M is operator of multiplication on a unimodular function from $C(K)$.

$A_n = \prod_{i=0}^{n-1} U^i A U^{-i} \in Z(X)$ and that $S^n f(k) = A_n(k) f(\varphi^n(k))$, $f \in C(K)$, $k \in K$. By Lemma 3.6 in [9] there is a point $k \in K$ such that

$$|A_n(k)| \geq 1 \text{ and } |A_n(\varphi^{-n}(k))| \leq 1, \quad n \in \mathbb{N}.$$

Consider two possibilities. (a) Point k is not φ -periodic. Let us fix ε , $0 < \varepsilon < 1$. Recalling that $\sigma(U) \subseteq \Gamma$ we see that there is $n \in \mathbb{N}$ such that

$$(1 - \varepsilon)^n < \varepsilon / (\|U^{-n}\| \|U^{n+1}\|). \quad (7)$$

Let V be a clopen neighborhood of k such that the sets $\varphi^j(V)$, $j \in \mathbb{Z}$, $|j| \leq n+1$ are pairwise disjoint and

$$|A_n(p)| \geq 1/2 \text{ and } |A_{n+1}(\varphi^{-n}(p))| \leq 2, \quad p \in V \quad (8)$$

Fix $y \in Y$ such that $\|y\| = 1$ and $\|(I - P_{\varphi^n(V)})y\| < \delta$, where δ will be chosen later, and consider $z = \sum_{j=0}^{2n} (1 - \varepsilon)^{|n-j|} T^j y$. Let $s = P_{\varphi^n(V)} y$

and $t = y - s$. Then $z = u + v$ where $u = \sum_{j=0}^{2n} (1 - \varepsilon)^{|n-j|} T^j s$ and $v = \sum_{j=0}^{2n} (1 - \varepsilon)^{|n-j|} T^j t$. Notice that $\|v\| \leq \delta \left(\sum_{j=0}^{2n} \|T^j\| \right)$. We proceed now to estimate the norm of u from below and the norm of $Tu - u$ from above. First notice that because the elements $T^j s$, $j = 0, \dots, 2n$ are pairwise disjoint we have $\|u\| \geq \|T^n s\|$. Next, $T^n s = A_n U^n s$ and therefore $\text{supp}(T^n s) \subseteq \text{supp}(U^n s) \subseteq \varphi^{-n}(\varphi^n(V)) = V$ whence, in view of (8), $\|T^n s\| \geq 1/2 \|U^n s\| \geq 1/2 \|s\| / \|U^{-n}\|$ and therefore

$$\|u\| \geq (1 - \delta) / (2 \|U^{-n}\|). \quad (9)$$

On the other hand the definition of u and pairwise disjointness of $T^j s$, $j = 0, \dots, 2n$ provides the following inequality

$$\|Tu - u\| \leq \varepsilon \|u\| + (1 - \varepsilon)^n \|s\| + \|(1 - \varepsilon)^n \|T^{2n+1} s\|. \quad (10)$$

Notice that (again in view of (8)) $\|T^{2n+1} s\| = \|T^{n+1} T^n s\| = \|A_{n+1} U^{n+1} T^n s\| \leq 2 \|U^{n+1}\| \|T^n s\| \leq 2 \|U^{n+1}\| \|u\|$. Combining the inequalities (7) and (10) we see that $\|Tu - u\| \leq 4\varepsilon \|u\|$. Next notice that

$$\|Tz - z\| \leq \|Tu - u\| + \|v\| + \|Tv\| \leq 4\varepsilon \|u\| + \delta(1 + \|T\|) \sum_{j=0}^{2n} \|T^j\|. \quad (11)$$

and

$$\|z\| \geq \|u\| - \|v\| \geq \|u\| - \delta \sum_{j=0}^{2n} \|T^j\|. \quad (12)$$

if we choose δ in such a way that

$$\delta < \frac{\varepsilon(1 - \delta)}{4\|U^{-n}\|(1 + \|T\|) \sum_{j=0}^{2n} \|T^j\|}$$

then the inequalities (9), (11), and (12) provide

$$\|Tz - z\| \leq 6\varepsilon\|u\|$$

and

$$\|z\| \geq \|u\|/2.$$

Finally, because ε can be chosen arbitrary small we see that $1 \in \sigma_{ap}(T, Y)$.

Now assume that point k is φ -periodic with the smallest period p . Then by Frolik's theorem [17, Theorem 6.25, page 150] there is a clopen neighborhood V of k which consists of φ -periodic points with period p . In this case $A_p(k) = 1$. For any $n \in \mathbb{N}$ we consider a clopen neighborhood V_n of k and $y_n \in Y$ with properties

- $V_n \subseteq V$.
- $\varphi^i(V_n) \cap \varphi^j(V_n) = \emptyset$, $0 \leq i < j \leq p-1$.
- $|A_p(t) - 1| \leq 1/n$, $t \in V_n$.
- $\|y_n\| = 1$.
- $\|y_n - P_n y_n\| \leq 1/n$ where P_n is the band projection corresponding to the set V_n .

Let $z_n = \sum_{j=0}^{p-1} T^j y_n$. Then $z_n = u_n + v_n$ where $u_n = \sum_{j=0}^{p-1} P_n T^j y_n$. We can

easily see that $\|v_n\| \leq \frac{1}{n} \sum_{j=0}^{p-1} \|T^j\|$ and that $\|u_n\| \geq \|P_n y_n\| \geq 1 - \frac{1}{n}$.

Therefore $\liminf_{n \rightarrow \infty} \|z_n\| \geq 1$. On the other hand $Tz_n - z_n = (Tu_n - u_n) + (Tv_n - v_n)$ and we already know that $Tv_n - v_n \xrightarrow{n \rightarrow \infty} 0$. Consider $Tu_n - u_n = T^p P_n y_n - P_n y_n = A_p U^p P_n y_n - P_n y_n$. We claim that $U^p P_n y_n = P_n y_n$. Indeed, if $f \in C(K)$ then $U^p P_n f U^{-p} = P_n f \circ \varphi^p = P_n f$ whence $U^p P_n f = P_n f U^p$. The center $Z(P_n X)$ of the band $P_n X$ can be identified with $P_n C(K)$ and therefore the operator $U^p P_n = P_n U^p P_n$ is a central operator on $P_n X$. Because this operator is also positive and its spectrum in $P_n X$ lies on the unit circle we see that $U^p P_n = P_n$. Therefore $Tu_n - u_n = A_p P_n y_n - P_n y_n \xrightarrow{n \rightarrow \infty} 0$ whence $1 \in \sigma_{ap}(T, Y)$.

Part 2. We will prove here that $\sigma_{ap}(T, X) \subseteq \sigma_{ap}(S, C(K))$. Assume that $\lambda \notin \sigma_{ap}(S, C(K))$. First consider $\lambda = 0$. Then clearly the multiplication operator A is invertible as an operator on $C(K)$ whence T is invertible on X . Thus we can assume that $\lambda = 1$. By [9, Theorem

3.29] and the cited above Frolik theorem the fact that $1 \notin \sigma_{ap}(S, C(K))$ implies that K is the union of disjoint φ -invariant sets K_1, K_2, K_3 and O with the properties

- I The sets K_1 and K_2 are closed in K , K_3 is clopen, and O is open.
- II There is $p \in \mathbb{N}$ such that $\varphi^p(k) = k, k \in K_3$.
- III $1 \notin \sigma(S, C(K_3))$.
- IV $\sigma(S, C(K_1)) \subseteq \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.
- V $\sigma(S, C(K_2)) \subseteq \{\lambda \in \mathbb{C} : |\lambda| > 1\}$.
- VI For any clopen subset E of O and for any clopen neighborhoods V_1, V_2 of K_1 and K_2 , respectively there is an $m \in \mathbb{N}$ such that $\varphi^m(E) \subseteq V_1$ and $\varphi^{-m}(E) \subseteq V_2$.

Assume contrary to our claim that there are $x_n \in X$ such that $\|x_n\| = 1$ and $Tx_n - x_n \xrightarrow{n \rightarrow \infty} 0$. It follows immediately from (II), (III) and $U \geq 0$ that $P_{K_3}x_n \rightarrow 0$. It also follows from (VI) that there is a clopen neighborhood V of K_1 such that $\varphi(V) \subseteq V$. Then S acts on $C(V)$ and by [9, Theorem 3.23] $\rho(T, C(V)) = \rho(T, C(K_1)) < 1$ where $\rho(T, X)$ is the spectral radius of T on X . Therefore there are $m \in \mathbb{N}$ and $a \in (0, 1)$ such that $\|A_p\|_{C(V)}\|U^p\| \leq a^p$ for $p \in \{m, m+1, \dots\}$. Let us fix such p . Then $T^p x_n - x_n \rightarrow 0$ whence $P_V T^p x_n - P_V x_n \rightarrow 0$. But $\|P_V T^p x_n\| = \|P_V A_p U^p x_n\| \leq a^p$ whence $\limsup \|P_V x_n\| \leq a^p$. Because p can be chosen arbitrary large we see that $\limsup \|P_V x_n\| = 0$. Therefore we can assume that $\text{supp } x_n \subseteq K_2 \cup (O \setminus V)$. But then it is not difficult to see from (V) and (VI) that there are $A > 1$ and $m \in \mathbb{N}$ such that $\|T^m x_n\| \geq A, \forall n \in \mathbb{N}$ in obvious contradiction with $T^m x_n - x_n \xrightarrow{n \rightarrow \infty} 0$.

Part 3. To finish the proof it remains to notice that the inclusion $\sigma_{ap}(T, Y) \subseteq \sigma_{ap}(T, X)$ is trivial. \square

Let now X be an arbitrary Banach lattice. If U is a d -isomorphism on X and $A \in Z(X)$ then A and U are order continuous operators on X and therefore by a result of Veksler [16] they can be extended in the unique way to order continuous operators \hat{A} and \hat{U} on the Dedekind completion \hat{X} of X . Moreover, $\hat{A} \in Z(\hat{X})$ and \hat{U} is a d -isomorphism of \hat{X} . It is not difficult to show using the order continuity of U and the definition of norm in \hat{X} that if $\sigma(U, X) \subseteq \Gamma$ then $\sigma(\hat{U}, \hat{X}) \subseteq \Gamma$. Notice also that Y is an almost localized subspace of X if and only if it is almost localized in \hat{X} . Let K and \hat{K} be the Gelfand compacts of $Z(X)$ and $Z(\hat{X})$, respectively. As above the maps $f \rightarrow UfU^{-1}, f \in Z(X)$ and $g \rightarrow \hat{U}g\hat{U}^{-1}, g \in Z(\hat{X})$ induce homeomorphisms φ of K and $\hat{\varphi}$ of

\hat{K} . We introduce the operators S on $C(K)$ and \hat{S} on $C(\hat{K})$ as

$$Sf = A(f \circ \varphi), f \in C(K)$$

and

$$\hat{S}f = A(f \circ \hat{\varphi}), f \in C(\hat{K}).$$

Surely X is almost localized in \hat{X} and therefore Theorem 0.19 and its proof provide the following corollary.

Corollary 0.20. *Let X be a Banach lattice, $A \in Z(X)$, U be a positive d -isomorphism of X , and $\sigma(U) \subseteq \Gamma$.*

Then $\sigma_{ap}(T, X) = \sigma_{ap}(\hat{T}, \hat{X}) = \sigma_{ap}(S, C(K)) = \sigma_{ap}(\hat{S}, C(\hat{K}))$.

But we can claim more.

Theorem 0.21. *Assume conditions of Corollary 0.20. Then*

$$\sigma(T, X) = \sigma(\hat{T}, \hat{X}) = \sigma(S, C(K)) = \sigma(\hat{S}, C(\hat{K}))$$

Proof. The proof will be divided into three steps.

Step 1. $\sigma(S, C(K)) = \sigma(\hat{S}, C(\hat{K}))$. First of all notice that the map $f \rightarrow \hat{f}$, $f \in Z(X)$ where \hat{f} is the unique extension of f on \hat{X} is an isomorphic and isometric embedding of $Z(X)$ into $Z(\hat{X})$. To this embedding corresponds the continuous map $\psi : \hat{K} \rightarrow K$. From the identity $\widehat{UfU^{-1}} = \hat{U}\hat{f}\hat{U}^{-1}$ we obtain that

$$\varphi(\psi(k)) = \psi(\hat{\varphi}(k)), k \in \hat{K}. \quad (13)$$

Assume now that $\lambda \in \sigma(\hat{S}, C(\hat{K})) \setminus \sigma(S, C(K))$. Then by Corollary 0.20 we have $\lambda \in \sigma_r(\hat{S}, C(\hat{K}))$. Clearly $\lambda \neq 0$ and it follows from Theorem 3.31 in [9] that there are a point $k \in \hat{K}$, $N \in \mathbb{N}$, and a positive ε such that

$$|\hat{A}_n(k)| \leq (|\lambda| - \varepsilon)^n \text{ and } |\hat{A}(\hat{\varphi}^{-n}(k))| \geq (|\lambda| + \varepsilon)^n, n \geq N. \quad (14)$$

Let $t = \psi(k)$ then it follows from (13) that

$$|A_n(t)| \leq (|\lambda| - \varepsilon)^n \text{ and } |A(\varphi^{-n}(t))| \geq (|\lambda| + \varepsilon)^n, n \geq N. \quad (15)$$

On the other hand it follows from the fact that $\lambda \notin \sigma(S, C(K))$ and Theorems 3.11 and 3.12 in [9] that K is the union of three disjoint φ -invariant sets K_1 , K_2 , and O such that

- (i) The sets K_1 and K_2 are closed subsets of K , $\sigma(S, C(K_1)) \subseteq \{\gamma \in \mathbb{C} : |\gamma| < |\lambda|\}$, and $\sigma(S, C(K_2)) \subseteq \{\gamma \in \mathbb{C} : |\gamma| > |\lambda|\}$.
- (ii) The set O is open in K , there is $p \in \mathbb{N}$ such that $\varphi^p(s) = s$, $s \in O$, and $\sigma(S, cl(O)) \subseteq \{\gamma \in \mathbb{C} : |\lambda| - \varepsilon/2 < |\gamma| < |\lambda| + \varepsilon/2\}$.

But the properties (i) and (ii) above clearly contradict (15) and thus we have proved that $\sigma(S, C(K)) \subseteq \sigma(\hat{S}, C(\hat{K}))$.

The inclusion $\sigma(\hat{S}, C(\hat{K})) \subseteq \sigma(S, C(K))$ can be proved in a similar way. Indeed, if $\lambda \in \sigma(S, C(K)) \setminus \sigma(\hat{S}, C(\hat{K}))$ then $\lambda \in \sigma_r(S, C(K))$ whence there are a point $t \in K$ and a positive ε satisfying (15). Let $k \in \psi^{-1}(t)$. Then in view of (13) we see that k and ε satisfy (14) in contradiction with our assumption that $\lambda \notin \sigma(\hat{S}, C(\hat{K}))$.

Step 2. $\sigma(\hat{T}, \hat{X}) = \sigma(\hat{S}, C(\hat{K}))$. Assume that $\lambda \notin \sigma(\hat{S}, C(\hat{K}))$. Then Theorems 3.11 and 3.12 in [9] and the Frolik's theorem guarantee that \hat{K} is the union of three disjoint, clopen, φ -invariant subsets \hat{K}_1 , \hat{K}_2 , and \hat{O} (some of them might be empty) such that

$$(\alpha) \sigma(\hat{S}, C(\hat{K}_1)) \subset \{\gamma \in \mathbb{C} : |\gamma| < |\lambda|\}.$$

$$(\beta) \sigma(\hat{S}, C(\hat{K}_2)) \subset \{\gamma \in \mathbb{C} : |\gamma| > |\lambda|\}.$$

$$(\gamma) \exists p \in \mathbb{N} \text{ such that } \hat{\varphi}^p(k) = k, k \in \hat{O}, \text{ and } \lambda \notin \sigma(\hat{S}, C(\hat{O})).$$

To the partition $(\hat{K}_1, \hat{K}_2, \hat{O})$ of the Stonean space \hat{K} corresponds the partition of \hat{X} into three pairwise disjoint bands \hat{X}_1 , \hat{X}_2 , and \hat{X}_O . These bands are \hat{U} -invariant because the sets $(\hat{K}_1, \hat{K}_2, \hat{O})$ are $\hat{\varphi}$ -invariant; and because \hat{A} is a band preserving operator they are \hat{T} -invariant as well. Consider the operator $\hat{T}_1 = \hat{T}|_{\hat{X}_1}$. Then $\|\hat{T}_1^n\| \leq \|\hat{A}_n|_{\hat{X}_1}\| \|\hat{U}^n\|$ whence, in view of (α) , $\rho(\hat{T}_1) \leq \rho(\hat{S}, C(\hat{K}_1))\rho(\hat{U}) < |\lambda|$. Next consider the operator $\hat{T}_2 = \hat{T}|_{\hat{X}_2}$. The condition (β) above guarantees that \hat{T}_2 is invertible and that $\rho(\hat{T}_2^{-1}) < |\lambda|$ whence $\sigma(\hat{T}_2, \hat{X}_2) \subset \{\gamma \in \mathbb{C} : |\gamma| > |\lambda|\}$. Finally let us consider the operator $\hat{T}_O = \hat{T}|_{\hat{X}_O}$. If we prove that $\lambda \notin \sigma(\hat{T}_O, \hat{X}_O)$ then it will follow that $\lambda \notin \sigma(\hat{T}, \hat{X})$. We assume that $\hat{O} \neq \emptyset$ as otherwise it is nothing to prove. By Frolik's theorem there are positive integers p_1, p_2, \dots, p_k such that $p_i \leq p, i = 1, \dots, k$ such that \hat{O} is the union of nonempty, clopen pairwise disjoint sets $\hat{O}_1, \dots, \hat{O}_k$ and the set \hat{O}_i consists of $\hat{\varphi}$ -periodic points of the smallest period p_i . Let Y_1, \dots, Y_i be the corresponding bands in \hat{X}_O . Then $\hat{T}^{p_i}|_{Y_i} = \hat{A}_{p_i}|_{Y_i}$ and it is immediate to see that

$$\sigma(\hat{T}, Y_i) = \sigma(\hat{S}, C(\hat{O}_i)) = \{\gamma \in \mathbb{C} : \exists t \in \hat{O}_i \text{ such that } \gamma^{p_i} = \hat{A}_{p_i}(t)\}. \quad (16)$$

In view of (16) and the condition (γ) we have $\lambda \notin \sigma(\hat{T}_O, \hat{X}_O)$ whence $\lambda \notin \sigma(\hat{T}, \hat{X})$ and thus we have proved that $\sigma(\hat{T}, \hat{X}) \subseteq \sigma(\hat{S}, C(\hat{K}))$.

Now assume that $\lambda \notin \sigma(\hat{T}, \hat{X})$. Then it follows from Theorem 12.20(1) in [2] that there is a band (maybe trivial) Y in \hat{X} such that $\hat{U}Y = Y$, $\exists p \in \mathbb{N}$ such that $\hat{\varphi}^p(t) = t$, $t \in \text{supp } Y$, and $\sigma(\hat{T}, Y^d) \cap \lambda\Gamma = \emptyset$. Then it follows from the proof of Theorem 13.9 in [2]

and the fact that $\hat{\varphi}$ is a homeomorphism of \hat{K} that Y^d is the direct sum of two disjoint (maybe trivial) T -invariant bands \hat{X}_1 and \hat{X}_2 such that $\sigma(\hat{T}|\hat{X}) \subset \{\gamma \in \mathbb{C} : |\gamma| < |\lambda|\}$ and $\sigma(\hat{T}|\hat{X}_2) \subset \{\gamma \in \mathbb{C} : |\gamma| > |\lambda|\}$. Applying the same kind of reasoning as above in the proof of the inclusion $\sigma(\hat{T}, \hat{X}) \subseteq \sigma(\hat{S}, C(\hat{K}))$ and based on the fact that $\sigma(\hat{U}) \subseteq \Gamma$ we see that $\lambda \notin \sigma(\hat{S}, C(\hat{K}))$.

Step 3. $\sigma(T, X) = \sigma(\hat{T}, \hat{X})$. The inclusion $\sigma(\hat{T}, \hat{X}) \subseteq \sigma(T, X)$ can be proved as in Theorem 0.14. Indeed, in the proof of Theorem 0.14 we did not use directly that \hat{X} has the weak Fatou property but only the statement of Theorem 0.13. Here we can instead refer to Theorem 0.19.

Assume that $\lambda \in \sigma(T, X) \setminus \sigma(\hat{T}, \hat{X})$. We can assume that $\lambda = 1$. Moreover by switching to factor operators like in the proof of Theorem 0.14 we can assume that $\Gamma \subset \sigma_r(T, X)$ and $\Gamma \cap \sigma(\hat{T}, \hat{X}) = \emptyset$. But then \hat{X} is the direct sum of two disjoint \hat{T} -invariant bands \hat{X}_1 and \hat{X}_2 such that $\sigma(\hat{T}, \hat{X}_1) \subseteq \{\alpha \in \mathbb{C} : |\alpha| < 1\}$ and $\sigma(\hat{T}, \hat{X}_2) \subseteq \{\alpha \in \mathbb{C} : |\alpha| > 1\}$. Let \hat{K}_1, \hat{K}_2 be the corresponding partition of \hat{K} . It follows from $\sigma(\hat{U}) \subseteq \Gamma$ that for some large enough $n \in \mathbb{N}$ we have $|\hat{A}_n| < 1$ on \hat{K}_1 and $|\hat{A}_n| > 1$ on \hat{K}_2 . Because $A_n \in Z(X)$ we conclude that X is the direct sum of two disjoint T -invariant bands X_1 and X_2 such that $\sigma(T, X_1) \subseteq \{\alpha \in \mathbb{C} : |\alpha| < 1\}$ and $\sigma(T, X_2) \subseteq \{\alpha \in \mathbb{C} : |\alpha| > 1\}$ in contradiction with our assumption that $\Gamma \subset \sigma(T, X)$. \square

We return now to almost localized subspaces of Banach lattices.

Theorem 0.22. *Assume conditions of Theorem 0.19. Then $\sigma(T, X) \subseteq \sigma(T, Y)$.*

Proof. By Theorem 0.21 we can assume that X is Dedekind complete. Assume that $\lambda \in \sigma(T, X) \setminus \sigma(T, Y)$. Then by Theorem 0.19 we have $\lambda \in \sigma_r(T, X)$. We can assume that $\lambda = 1$. Combining Theorems 0.21 and 0.19 with Theorems 3.29 and 3.31 from [9] we see that there is a band B in X such that its Banach dual B^* has the following properties. The bands $(S^*)^{-i}B(S^*)^i, i \in \mathbb{Z}$, are pairwise disjoint.

For any $F_0 \in B^*$ there are a sequence $\{F_i \in X^*, i \in \mathbb{Z}\}$, and $\varepsilon > 0$ such that $T^*F_i = F_{i-1}, i \in \mathbb{Z}$, and the vector function $G(\alpha) = \sum_{i=-\infty}^{\infty} \alpha^i F_i$ is analytic in the annulus $\{1 - \varepsilon < |\alpha| < 1 + \varepsilon\}$.

Let $y \in B$ be such that $\|y\| = 1$ and $\|P_B y\| > 3/4$. Then there is an $F \in B^*$ such that $\langle y, F \rangle \geq 1/2$. On the other hand $\langle y, G(\alpha) \rangle \equiv 0$ in some open neighborhood of Γ whence $\langle y, F \rangle = 0$, a contradiction. \square

Corollary 0.23. *Assume conditions of Theorem 0.19. Assume additionally that Y is an analytic subspace of X and that $U^n \neq I$ for any $n \in \mathbb{N}$. Then $\sigma(T, Y)$ is a connected rotation invariant subset of \mathbb{C} .*

Proof. In virtue of previous results we can assume without loss of generality that X is Dedekind complete. Let us show first that $\sigma(T, Y)$ is rotation invariant. Indeed, otherwise there are a nonzero band B in X and a natural n such that $(T^n - A_n)B = 0$. Because Y is an analytic subspace of X it follows that $(T^n - A_n)Y = 0$. But Y is almost localized in X whence $\varphi^n(k) = k$, $k \in K$. Therefore $U^n f = f U^n$, $f \in Z(X)$ and thus $U^n \in Z(X)$. But $U \geq 0$ and $\sigma(U) \subseteq \Gamma$ whence $U^n = I$ in contradiction with our assumption.

It remains to prove that the set $\{|\lambda| : \lambda \in \sigma(T, Y)\}$ is connected. Assume to the contrary that there is a positive r such that Y is the direct sum of two nontrivial spectral subspaces Y_1 and Y_2 , $\sigma(T, Y_1) \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$, and $\sigma(T, Y_2) \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$. It follows from Theorems 0.19 and 0.22, from the fact that $\sigma_r(T, Y)$ is open in \mathbb{C} , and from Theorem 13.1 in [2] that X is the direct sum of two T -invariant bands X_1 and X_2 such that $\sigma(T, X_1) \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and $\sigma(T, X_2) \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$. Then obviously $Y_i \subseteq X_i$, $i = 1, 2$ in contradiction with Y being an analytic subspace of X . \square

Let us consider some examples.

Example 0.24. Let Γ^n be the n -torus and m be the normalized Lebesgue measure on Γ^n . Let X be a Banach space of m -measurable functions on Γ^n such that $L^\infty(\Gamma^n, m) \subseteq X \subseteq L^1(\Gamma^n, m)$ and the norm in X is rotation invariant. Let Y be the (closed) subspace of X consisting of all functions from X analytic in the open polydisk \mathbb{D}^n . Finally, let U be an operator on Y induced by a non periodic rotation on Γ^n , $A \in H^\infty(\Gamma^n, m)$, and $T = AU$.

Then $\sigma(T, Y)$ is a connected rotation invariant subset of \mathbb{C} .

Moreover, if U is generated by a strictly ergodic rotation of Γ^n then $\sigma(T, Y)$ is a circle centered at 0 if A is invertible in H^∞ and a disk (or point 0) otherwise.

Example 0.25. This example is similar to Example 0.24 but instead of polydisk we consider the unit ball B^n in \mathbb{C}^n , norm in X is invariant under unitary transformations of B^n and operator U is composition operator generated by a non periodic unitary transformation of B^n .

Example 0.26. Let m be the normalized Lebesgue measure on Γ and $p \in (1, \infty)$. Let X be a rearrangement invariant Banach space of m -measurable functions such that its lower and upper Boyd indices (see

e.g. [11, Section 2(b)]) are both equal to p . An example of such a space is the Lorentz space Λ^p . Let φ be a Mobius transformation of the unit disk D . Let V be the composition operator on X generated by φ , A be an invertible function from the disk algebra $A(D)$, and $T = AV$. Let Y be the closed subspace of X which consists of all functions from X that are boundary values of functions analytic in D . Then T is bounded on X and

- (a) If φ is a non periodic elliptic transformation then $\sigma(T, Y) = A(\xi)\Gamma$ where ξ is the fixed point of φ in D .
- (b) If φ is a parabolic transformation then $\sigma(T, Y) = A(\xi)\Gamma$ where ξ is the fixed point of φ on Γ .
- (c) If φ is a hyperbolic transformation then $\sigma(T, Y)$ is the annulus with radii $|A(\xi_1)|\varphi'(\xi_1)|^{-1/p}$ and $|A(\xi_2)|\varphi'(\xi_2)|^{-1/p}$ where ξ_1 and ξ_2 are fixed points of φ on Γ .

Proof. Let $U = |\varphi'|^{1/p}V$. It is enough to prove that $\sigma(U, X) \subseteq \Gamma$. Indeed, after it proved we can apply our previous results and the formula for spectral radius of weighted composition operators in $C(K)$ (see e.g. [9, Theorem 3.23]). By Boyd's theorem (see [11] or the original Boyd's paper [6]) for any ε , $0 < \varepsilon < p - 1$, X is an interpolation space between $L^{p-\varepsilon}$ and $L^{p+\varepsilon}$ and the constant of interpolation does not depend on ε . The operator U is bounded on $L^{p-\varepsilon}$ and on $L^{p+\varepsilon}$ whence it is bounded on X . Next notice that $U = |\varphi'|^{\frac{\varepsilon}{p(p-\varepsilon)}}W_{1,\varepsilon}$ and $U = |\varphi'|^{\frac{\varepsilon}{p(p+\varepsilon)}}W_{2,\varepsilon}$ where $W_{1,\varepsilon}$ and $W_{2,\varepsilon}$ are isometries in $L^{p-\varepsilon}$ and $L^{p+\varepsilon}$, respectively. Applying Theorem 0.19 and Boyd's interpolation theorem we see that $\sigma(U, X) \subseteq \Gamma$. \square

Let us return to the general case when we can assume the conditions of Theorem 0.19. Simple examples show that in general $\sigma(T, X) \subsetneq \sigma(T, Y)$. Therefore it is natural to ask what can be said about the set $\sigma(T, Y) \setminus \sigma(T, X)$. Assume in addition to the conditions of Corollary 0.20 that $UY = Y$. Then $AY \subseteq Y$. Let $Z(X)$ be the ideal center of X . Let $\mathcal{A} = \{f \in Z(X) : fY \subseteq Y\}$. Then \mathcal{A} is a unital closed subalgebra of $Z(X)$. We will denote its space of maximal ideals by \mathfrak{M} and its Shilov boundary by ∂ . We will identify an element $a \in \mathcal{A}$ and its Gelfand image. Notice that the map $a \rightarrow U^{-1}aU$ is an automorphism of \mathcal{A} . Let ϕ be the corresponding homeomorphism of \mathfrak{M} . We can consider weighted composition operator $Sf = a(f \circ \phi)$ either on $C(\mathfrak{M})$ or on $C(\partial)$.

Lemma 0.27. *Let $a \in \mathcal{A}$*

- (1) $\|a\|_{Z(X)} = \|a\|_{\mathcal{L}(Y)}$ and
- (2) a is invertible in \mathcal{A} if and only if it is invertible in $\mathcal{L}(Y)$.

Proof. (1) Clearly $\|a\|_{\mathcal{L}(Y)} \leq \|a\|_{\mathcal{L}(X)} = \|a\|_{Z(X)}$. To prove the converse inequality let us assume that $\|a\|_{Z(X)} = 1$ and take $\varepsilon > 0$. Let K be the Stonean compact of \hat{X} . Recall that $Z(X)$ is isometrically embedded into $C(K)$. Let $E = cl\{k \in K : |a(k)| > 1 - \varepsilon\}$ and let B be the band in \hat{X} corresponding to E . Because Y is almost localized in X there is $y \in Y$ such that $\|y\| = 1$ and $\|(I - P_B)y\|_{\hat{X}} < \varepsilon$. Then $\|ay\| \geq (1 - \varepsilon)^2$ whence $\|a\|_{\mathcal{L}(Y)} = 1$.

(2). Clearly, if a is invertible in \mathcal{A} then it is invertible in $\mathcal{L}(Y)$. Assume that a is invertible in $\mathcal{L}(Y)$. It follows easily from the fact that Y is almost localized in X that a is invertible in $Z(X)$ (indeed, otherwise we can for any $\varepsilon > 0$ find a $y \in Y$ such that $\|y\| = 1$ but $\|ay\| < \varepsilon$). Let b be the inverse of a in $\mathcal{L}(Y)$ and c its inverse in $Z(X)$. Then for any $y \in Y$ we have $(b - c)ay = 0$, but $aY = Y$ whence $cY \subseteq Y$ and a is invertible in \mathcal{A} . \square

Proposition 0.28. *Assume conditions of Theorem 0.19 and that $UY = Y$. Then*

$$|\sigma(T, Y)| = |\sigma(S, C(\mathfrak{M}))|.$$

Proof. Assume that $r \notin |\sigma(T, Y)|$. Notice that $\rho(T, Y) = \rho(S, C(\mathfrak{M}))$. By Lemma 0.27 the operators T and S are either both invertible or both not invertible; and if they are both invertible then $\rho(T^{-1}) = \rho(S^{-1})$. Therefore let us consider the case when Y is the direct sum of nonzero spectral subspaces Y_1 and Y_2 , $\sigma(T, Y_1) \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$, and $\sigma(T, Y_2) \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$. Then by Theorem 0.22 and by Theorem 13.1 in [2] X is the sum of two disjoint T -invariant bands X_1 and X_2 , $\sigma(T, X_1) \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$, and $\sigma(T, X_2) \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$. Let P be the band projection on the band X_1 . Then $PY = Y_1$ whence $P \in \mathcal{A}$. By the Shilov idempotent theorem \mathfrak{M} is the union of two clopen sets \mathfrak{M}_1 and \mathfrak{M}_2 . It is routine to verify that these sets are ϕ -invariant and that $\sigma(S, C(\mathfrak{M}_1)) \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and $\sigma(S, C(\mathfrak{M}_2)) \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$.

Conversely, if $\sigma(S, C(\mathfrak{M})) = \sigma_1 \cup \sigma_2$ where $\sigma_1 \subset \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and $\sigma_2 \subset \{\lambda \in \mathbb{C} : |\lambda| > r\}$ then (see [9]) \mathfrak{M} is the union of two clopen ϕ -invariant sets \mathfrak{M}_1 and \mathfrak{M}_2 and $\sigma(S, C(\mathfrak{M}_i)) = \sigma_i, i = 1, 2$. Therefore the spectral projections P_1 and $P_2 = I - P_1$ corresponding to σ_1 and σ_2 belong to \mathcal{A} . Let $Y_i = P_i Y, i = 1, 2$, then it is immediate to see that $TY_i \subseteq Y_i, i = 1, 2$ and that $|\sigma(T|Y_1)| \subset (0, r), |\sigma(T|Y_2)| \subset (r, \infty)$. \square

We will just outline the proof of the following proposition.

Proposition 0.29. *Assume conditions of Proposition 0.28. Then $\sigma(S, C(\mathfrak{M})) \subseteq \sigma(T, Y)$.*

Sketch of the proof. We have to consider three possibilities.

- (1) $\lambda \in \sigma_{ap}(S, C(\mathfrak{M}))$. Applying part (1) of Theorem 4.2 in [9], Lemma 3.6 from [9] and Theorem 0.21 we conclude that $\lambda \in \sigma_{ap}(S, C(\mathfrak{M}))$.
- (2) $\lambda \in \sigma_r(S, C(\mathfrak{M})) \cap \sigma_r(C(\partial))$. We apply Theorem 3.26 from [9] and Theorem 0.21 to conclude that $\lambda \in \sigma_r(T, Y)$.
- (3) $\lambda \in \sigma_r(S, C(\mathfrak{M})) \setminus \sigma(S, C(\partial))$. In this case if $\lambda \notin \sigma(T, Y)$ then $\lambda\Gamma \cap \sigma(T, Y) = \emptyset$ in contradiction with Proposition 0.28. \square

Propositions 0.28 and 0.29 beg the question whether $\sigma(T, Y) = \sigma(S, C(\mathfrak{M}))$. Without additional assumptions it might be false. Indeed (see [18]) there are a Banach lattice X with trivial center and an isometry U of X such that $U^n \neq I, n \in \mathbb{N}$.⁴ Then $\sigma(U, X) = \Gamma$ but $\sigma(S, C(\mathfrak{M})) = \{1\}$.

Therefore if we want the equality $\sigma(T, Y) = \sigma(S, C(\mathfrak{M}))$ to be true we must assume some "richness" of the algebra \mathcal{A} .

Conjecture 0.30. Assume conditions of Proposition 0.28 and assume additionally that \mathcal{A} is almost localized in $C(K)$ where K is the Stonean compact of X . Then

- (1) $\sigma(T, Y) = \sigma(S, C(\mathfrak{M}))$.
- (2) $\sigma_{ap}(T, Y) = \sigma_{ap}(S, C(\partial))$.

Remark 0.31. There are two arguments in favor of Conjecture 0.30 to be correct.

First, it is true when Y is a unital uniform algebra (see [9, Theorem 3.26]).

Second, it is not difficult to prove that it is true if the algebra \mathcal{A} is analytic in $C(K)$.

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⁴ I am grateful to A. W. Wickstead for information about the corresponding example.

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COMMUNITY COLLEGE OF PHILADELPHIA, 1700 SPRING GARDEN ST., PHILADELPHIA, PA, USA

E-mail address: akitover@ccp.edu